Problem 1.22.4 Consider the following equation for an unknown function $f : [0,1] \to \mathbb{R}$:

$$f(x) = g(x) + \lambda \int_0^1 (x-y)^2 f(y)\, dy + \frac{1}{2}\sin(f(x)).$$

Prove that there exists a number $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0)$, and all continuous functions g on $[0,1]$, the equation has a unique continuous solution.

Problem 1.23.2 Can one find a bounded sequence of real numbers $\{x_n\}, n \in \mathbb{Z}$ that satisfies

$$x_n = \sin(n) + 0.5x_{n-1} + 0.4\sin(x_{n+1})$$

for every $n \in \mathbb{Z}$?

Problem 1.21.5 Let K be a continuous function on $[0,1]\times[0,1]$ satisfying $|K| < 1$. Suppose that g is a continuous function on $[0,1]$. Show that there exists a unique continuous function f on $[0,1]$ such that

$$f(x) = g(x) + \int_0^1 f(y)K(x,y)\,dy.$$

Rubric:
+**1 point** for identifying, in any way, that the contraction mapping principle / Banach fixed point theorem must be used.
+**2 points** for identifying an appropriate self-map. (award at most +**1 point** if it is not specified anywhere that the chosen metric space of the map is, indeed, a complete metric space.)
+**2 points** for justifying that the output of the mapping is an element of the claimed complete metric space.
+**4 points** for fully justifying that the mapping is a contraction mapping; **-1 point** if $d(Tf, Tg) < d(f, g)$ is shown but not $d(Tf, Tg) \leq qd(f, g)$ for some constant $0 < q < 1$.
+**1 point** for a correct conclusion about a unique fixed point, and attributing it to the Banach fixed point theorem or contraction mapping principle.